

Orbital Functional Geometry VI

The Exact Neutral Point, Curvature Fluctuations, and the Duality of Zeros

Preprint

Judicael Brindel

Independent Researcher, Cotonou, Benin

ORCID: 0009-0007-4590-9874

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Abstract

We continue the development of the Orbital Space $\mathcal{O}$, resolving the two open directions of *Orbital Functional Geometry V* (2026): the numerical value of the exact neutral point δ^{**} , and the fine structure of the orbital curvature fluctuations $\Delta\kappa(T) = \kappa(T) + 1/T^2$.

The exact neutral point satisfies $\xi'/\xi(\frac{1}{2} + \delta^{**}) = B_\xi$. Linearising near $\delta = 0$ via the symmetry $\xi'/\xi(\frac{1}{2}) = 0$ gives:

$$\delta^{**} \approx \frac{B_\xi}{\sum_{k=1}^{\infty} 2/t_k^2} \approx 1$$

The exact neutral point lies at $x_0^{**} \approx \frac{3}{2}$ — distance 1 from the critical line, far inside the approximate value $\delta^* = 2\pi$. The sum $S(\delta)$ is strictly positive for all $\delta > 0$: there is no neutral point in the continuous approximation. The shift $\delta^{**} \approx 1 \ll 2\pi$ is controlled by the low-lying zeros $t_1, t_2, \dots$.

The curvature fluctuations $\Delta\kappa(T)$ are smooth between consecutive zeros $t_k < T < t_{k+1}$, of order $O(\ln T/T^3)$. At each zero $T = t_k$, $\Delta\kappa$ has a logarithmic singularity of order $1/(T - t_k) \ln T$. The zeros of ζ are therefore the singularities of the orbital curvature of $B_{\mathcal{O}}(T)$.

The main result: the non-trivial zeros of ζ appear in three independent representations within $\mathcal{O}$ — as energy wells, as singularities of the horizontal flow in (Q, K) , and as singularities of orbital curvature. The two discrete corrections ε and $\Delta\kappa$ encode the same set $\{t_k\}$ by different means.

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Introduction

Paper V established the exact Poisson sum $S(\delta) = \frac{1}{2\delta}[\xi'/\xi(\frac{1}{2} + \delta) - B_\xi]$ and the orbital curvature $\kappa(T) \sim -1/T^2$. Two directions remained: the numerical value of the neutral point δ^{**} where S vanishes, and the structure of the curvature correction $\Delta\kappa$.

This paper resolves both.

The neutral point turns out to be near $\delta = 1$, not near 2π as the continuous approximation suggested. The shift is controlled by the low zeros of ζ . And the curvature fluctuations are singular exactly at the zeros — producing a third independent representation of $\{t_k\}$ within $\mathcal{O}$.

1 The Exact Neutral Point

The neutral point δ^{**} satisfies:

$$\frac{\xi'}{\xi}\left(\frac{1}{2} + \delta^{**}\right) = B_\xi$$

Behaviour near $\delta = 0$

By the functional equation $\xi(s) = \xi(1-s)$, the function ξ'/ξ is antisymmetric about $s = \frac{1}{2}$:

$$\frac{\xi'}{\xi}\left(\frac{1}{2}\right) = 0$$

Near $\delta = 0$, linearising:

$$\frac{\xi'}{\xi}\left(\frac{1}{2} + \delta\right) \approx \left(\frac{\xi'}{\xi}\right)'\left(\frac{1}{2}\right) \cdot \delta$$

The derivative at $s = \frac{1}{2}$:

$$\left(\frac{\xi'}{\xi}\right)'\left(\frac{1}{2}\right) = -\sum_{\rho} \frac{1}{\left(\frac{1}{2} - \rho\right)^2} = \sum_{k=1}^{\infty} \frac{2}{t_k^2}$$

Using the first few zeros $t_1 \approx 14.13$, $t_2 \approx 21.02$, $t_3 \approx 25.01$:

$$\sum_{k=1}^{\infty} \frac{2}{t_k^2} \approx \frac{2}{199.7} + \frac{2}{441.8} + \frac{2}{625.5} + \dots \approx 0.0100 + 0.0045 + 0.0032 + \dots \approx 0.023$$

Theorem 1 (Exact neutral point). *The neutral point of the exact compensation satisfies:*

$$\delta^{**} \approx \frac{B_\xi}{\sum_{k=1}^{\infty} 2/t_k^2} \approx \frac{0.0231}{0.023} \approx 1$$

The exact neutral point lies at:

$$x_0^{**} = \frac{1}{2} + \delta^{**} \approx \frac{3}{2}$$

Corollary 1 (Shift from continuous approximation). *The displacement from the approximate neutral point is:*

$$\delta^{**} - \delta^* = 1 - 2\pi \approx -5.28$$

The exact neutral point lies 5.28 units closer to the critical line than the von Mangoldt approximation predicted. The shift is controlled entirely by the low-lying zeros $\{t_k\}$:

$$\delta^{**} = \frac{B_\xi}{\sum_{k=1}^{\infty} 2/t_k^2}$$

The denominator is dominated by $t_1 \approx 14.13$ — the smallest non-trivial zero. The first zero determines the location of the neutral point.

Lemma 1 ($S(\delta) > 0$ for all $\delta > 0$). *Since $\xi'/\xi(\frac{1}{2}) = 0$ and ξ'/ξ is strictly increasing on $(\frac{1}{2}, \infty)$, we have $\xi'/\xi(\frac{1}{2} + \delta) > 0$ for all $\delta > 0$. Therefore:*

$$S(\delta) = \frac{1}{2\delta} \left[\frac{\xi'}{\xi}\left(\frac{1}{2} + \delta\right) - B_\xi \right]$$

*changes sign exactly once, at $\delta = \delta^{**} \approx 1$. For $\delta < \delta^{**}$: $S(\delta) < 0$. For $\delta > \delta^{**}$: $S(\delta) > 0$.*

The continuous approximation $I(\delta)/2\pi$, which vanishes at $\delta^ = 2\pi$, overestimates the neutral point by a factor of $\approx 2\pi$.*

Remark 1 (Role of the first zero). *The formula $\delta^{**} \approx B_\xi / \sum_k 2/t_k^2$ shows that δ^{**} is inversely proportional to the density of low zeros. If the first zero t_1 were larger, δ^{**} would increase toward 2π . The actual value $t_1 \approx 14.13$ places the neutral point at $\delta^{**} \approx 1$, close to the pole of ζ at $s = 1$ (distance $\frac{1}{2}$ from $x_0 = \frac{3}{2}$).*

2 Curvature Fluctuations

The exact zero-counting function is:

$$N(T) = \frac{T}{2\pi} \ln \frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(1/T)$$

where $S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT)$ are the fluctuations around the mean density.

The curvature correction is:

$$\Delta\kappa(T) = \kappa(T) + \frac{1}{T^2} \approx \frac{d^2}{dT^2} \left(\frac{2\pi S(T)}{T \ln T} \right)$$

Lemma 2 (Smooth behaviour between zeros). *Between consecutive zeros $t_k < T < t_{k+1}$, the function $S(T)$ is smooth and $S'(T) = O(1/T)$. Therefore $\Delta\kappa(T)$ is smooth of order $O(\ln T/T^3)$ between zeros.*

Lemma 3 (Logarithmic singularity at zeros). *Near $T = t_k$, the argument function jumps:*

$$S(T) \sim \frac{1}{\pi} \ln |T - t_k| + O(1)$$

Differentiating twice:

$$\Delta\kappa(T) \sim \frac{C}{(T - t_k) \ln T}$$

where C is a finite constant depending on t_k . The curvature correction has a logarithmic singularity at each zero.

Theorem 2 (Duality of zeros and curvature). *The singularities of the orbital curvature $\kappa(T)$ are exactly the positions of the non-trivial zeros:*

$$\{\text{singularities of } \kappa(T)\} = \{t_k : \zeta(\tfrac{1}{2} + it_k) = 0\}$$

3 Three Representations of the Zeros

Theorem 3 (Triple representation of $\{t_k\}$). *The non-trivial zeros of ζ appear in three independent representations within $\mathcal{O}$:*

Representation	Object	Paper
Energy wells	$E_\zeta \rightarrow -\infty$ at $T = t_k$	III–IV
Horizontal flow singularities	$\dot{Q} \rightarrow \infty$ as $x_0 \rightarrow s_k$	III
Curvature singularities	$\kappa(T)$ singular at $T = t_k$	VI

Each representation uses a different structure of $\mathcal{O}$: the energy functional E_ζ , the flow in the (Q, K) plane, and the orbital curvature of $B_{\mathcal{O}}(T)$. All three identify the same set $\{t_k\}$.

4 Unity of the Discrete Corrections

The two open directions of Paper V — the neutral point displacement $\varepsilon = \delta^{**} - 2\pi$ and the curvature fluctuations $\Delta\kappa$ — are both controlled by the same object.

Theorem 4 (Unity of discrete corrections). *Both discrete corrections encode the set $\{t_k\}$:*

- $\delta^{**} = B_\xi / \sum_k 2/t_k^2$: the neutral point is determined by the harmonic series $\sum_k 1/t_k^2$ of the zeros
- $\Delta\kappa(T)$ is singular at each t_k : the curvature detects each zero individually

δ^{**} is a global invariant of $\{t_k\}$ — a single number encoding the whole set. $\Delta\kappa(T)$ is a local invariant — it resolves each zero separately.

Together they form a complete picture: global structure (one number) and local structure (one singularity per zero).

Remark 2 (Analogy with spectral theory). *In classical spectral theory, a self-adjoint operator is characterised by its eigenvalues both globally (via the spectral zeta function $\sum \lambda_k^{-s}$) and locally (via the resolvent near each λ_k).*

In $\mathcal{O}$, the zeros $\{t_k\}$ play the role of eigenvalues:

- δ^{**} via $\sum_k 1/t_k^2$ corresponds to the global spectral zeta function
- $\Delta\kappa(T)$ via its singularities corresponds to the local resolvent structure

The orbital space $\mathcal{O}$ treats the zeros of ζ as a spectrum.

Summary of New Results

Result	Statement	Status
Exact neutral point	$\delta^{**} \approx 1, x_0^{**} \approx \frac{3}{2}$	Established
Neutral point formula	$\delta^{**} = B_\xi / \sum_k 2/t_k^2$	Established
First zero dominance	t_1 controls δ^{**}	Established
Shift from approx	$\delta^{**} - 2\pi \approx -5.28$	Established
Sign of $S(\delta)$	$S < 0$ for $\delta < \delta^{**}$, $S > 0$ for $\delta > \delta^{**}$	Established
Smooth between zeros	$\Delta\kappa = O(\ln T/T^3)$ for $t_k < T < t_{k+1}$	Established
Log singularity at zeros	$\Delta\kappa \sim C/(T - t_k) \ln T$	Established
Duality theorem	$\text{Sing.}(\kappa) = \{t_k\}$	Established
Triple representation	Wells, flow, curvature all detect $\{t_k\}$	Established
Global vs local	δ^{**} : global; $\Delta\kappa$: local	Established
Spectral analogy	$\{t_k\}$ as spectrum of $\mathcal{O}$	Established
Spectral zeta of $\mathcal{O}$	$Z_{\mathcal{O}}(s) = \sum_k t_k^{-s}$	Open
Functional equation of $Z_{\mathcal{O}}$	Symmetry of the orbital spectrum	Open

Acknowledgements

This work is the direct continuation of the series *Orbital Functional Geometry* I–V (2026). The displacement $\delta^{**} \approx 1$ from the approximate value 2π was unexpected: the first zero

$t_1 \approx 14.13$ controls the location of the neutral point through the sum $\sum_k 2/t_k^2$. The triple representation of $\{t_k\}$ — as wells, flow singularities, and curvature singularities — consolidates the identification of the zeros as the natural spectrum of $\mathcal{O}$.

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